



Monica Neagoy is the Program Director of the Division of Elementary, Secondary, and Informal Education at the National Science Foundation. She earned her Bachelor of Science in pure mathematics from Georgetown University, her M.A. from the Catholic University of America, and her Ph.D. in mathematics education from the University of Maryland. Professor Neagoy has been teaching mathematics and mathematics education at Georgetown University for more than 15 years.

Guidebook Contents

Parts 1, 2, and 3

- | | |
|--|---|
| Lesson 1: An Overview | Lesson 17: Modeling Real-World Data with Quadratic Functions |
| Lesson 2: The Language of Algebra | Lesson 18: Polynomial Explorations (Degree Greater than Two) |
| Lesson 3: Exploring Functions with the Aid of Graphing Calculators | Lesson 19: Rational Functions—Introductory Explorations |
| Lesson 4: Linear Functions—Introductory Explorations | Lesson 20: The Geometry of Rational Function Graphs |
| Lesson 5: Multiple Representations of Linear Functions | Lesson 21: Working with Rational Functions and Equations |
| Lesson 6: The Geometry of Linear Function Graphs | Lesson 22: Exponential Functions—Introductory Explorations |
| Lesson 7: Words, Equations, Numbers, and Graphs | Lesson 23: The Geometry of Exponential Function Graphs |
| Lesson 8: Problem Solving with Linear Equations | Lesson 24: Working with Exponential Functions and Equations |
| Lesson 9: Modeling Real-World Data with Linear Functions | Lesson 25: Systems of Linear Functions and Equations |
| Lesson 10: Linear Functions and Geometry | Lesson 26: Using Matrices to Solve Linear Systems |
| Lesson 11: Quadratic Functions—Introductory Explorations I | Lesson 27: Systems of Functions and Equations |
| Lesson 12: Quadratic Functions—Introductory Explorations II | Lesson 28: Systems of Inequalities |
| Lesson 13: The Geometry of Quadratic Function Graphs | Lesson 29: Iterating Functions—Looking at Functions Recursively |
| Lesson 14: Words, Equations, Numbers, and Graphs | Lesson 30: Using Iteration as a Problem Solving Tool |
| Lesson 15: Problem Solving with Quadratic Equations | |

THE TEACHING COMPANY®
4151 Lafayette Center Drive, Suite 100
Chantilly, VA 20151-1232
Phone: 1-800-TEACH-12 (1-800-832-2412)
Fax: 703-378-3819
www.TEACH12.com

Cover image: Earth Enveloping the Sun.
© Digital Art/CORBIS

©1997 The Teaching Company.



Algebra I

Taught by: Monica Neagoy
Georgetown University

Parts 1, 2, and 3

Monica Neagoy, Ph.D.

Georgetown University

Dr. Monica Neagoy has been a professional lecturer in Mathematics at Georgetown University since 1980. She essentially designs, directs, and teaches institutes for teachers and students of mathematics. In addition, Dr. Neagoy serves the community as a mathematics/mathematics education consultant on a wide variety of mathematical content and method areas. Countless private schools and public school systems are listed among her clients, as well as private organizations such as The Annenberg Foundation, Public Broadcasting Systems, The Cafritz Foundation, and The International Satellite Corporation (INTELSAT). Finally, she is highly esteemed in the mathematical community on both a national and international level: she is one of seven national judges for the prestigious MATHCOUNTS organization, and is frequently invited to speak—often to give the keynote address—at conferences and special mathematics events in the United States and abroad.

On another note, Dr. Neagoy co-directs the professional LE NEON French-American Theater in Arlington, Virginia where she does everything from acting and directing to fund raising and public relations. Also on the “artistic” side, she has made her first film, *Five Days in Paris*, which premiered in April 1994 at the Kennedy Center for the Performing Arts as a part of FILMIEST DC. Dr. Neagoy had the lead role in this film.

Table of Contents

Algebra I Parts I, II, and III

Professor Biography	i
---------------------------	---

Section I: Exploring Algebra

Lesson One	An Overview	1
Lesson Two	The Evolution of Numbers	2
Lesson Three	The Language of Algebra	3
Lesson Four	Exploring Functions with the Aid of Graphing Calculators	4

Section II: Exploring Linear Functions

Lesson Five	Linear Functions—Introductory Explorations	5
Lesson Six	Multiple Presentations of Linear Functions	6
Lesson Seven	The Geometry of Linear Function Graphs	7
Lesson Eight	Word, Equations, Numbers, and Graphs	8
Lesson Nine	Problem Solving with Linear Equations	9
Lesson Ten	Modeling Real-World Data with Linear Functions	10
Lesson Eleven	Linear Functions and Geometry	11

Section III: Exploring Quadratic Equations

Lesson Twelve	Quadratic Functions— Introductory Explorations I	12
Lesson Thirteen	Quadratic Functions— Introductory Explorations II	13
Lesson Fourteen	The Geometry of Quadratic Function Graphs	14
Lesson Fifteen	Words, Equations, Numbers, and Graphs	15
Lesson Sixteen	Problem Solving with Quadratic Equations	16
Lesson Seventeen	Modeling Real-World Data with Quadratic Functions	17
Lesson Eighteen	Polynomial Explorations (Degree Greater than Two)	18

Section IV: Exploring Rational Functions

Lesson Nineteen	Rational Functions—Introductory Explorations	19
Lesson Twenty	The Geometry of Rational Function Graphs	20
Lesson Twenty-One	Working with Rational Functions and Equations ...	21

Section V: Exploring Exponential Functions

Lesson Twenty-Two	Exponential Functions— Introductory Explorations	22
Lesson Twenty-Three	The Geometry of Exponential Function Graphs	23
Lesson Twenty-Four	Working with Exponential Functions and Equations	24

Section VI: Solving Systems of Equations and Inequalities

Lesson Twenty-Five	Systems of Linear Functions and Equations	25
Lesson Twenty-Six	Using Matrices to Solve Linear Systems	26
Lesson Twenty-Seven	Systems of Functions and Equations	27
Lesson Twenty-Eight	Systems of Inequalities	28

Section VII: Iterating Functions

Lesson Twenty-Nine	Iterating Functions—Looking at Functions Recursively	29
Lesson Thirty	Using Iteration as a Problem Solving Tool	30

Lesson One

An Overview

- I.** Introductory Remarks
- II.** An Overview of Algebra
 - A.** The Origin of the word “algebra”
 - B.** The meaning of algebra
 - C.** The Development of Algebraic Symbolism
 - 1.** Rhetorical phase
 - 2.** Syncopated phase
 - 3.** Symbolic phase
 - D.** The geometric and numeric currents
- III.** An Overview of this course
 - A.** Topics to receive increased attention
 - B.** Topics to receive decreased attention
 - C.** Reason for the shifts in perspective: Technology
 - D.** Outline of the course: Seven Sections
 - E.** Importance of the accompanying workbook
 - F.** History: A humanizing effect
 - G.** A variety of media
 - H.** An integrated approach
 - I.** Action verbs
- IV.** A fun problem: “Think of a number”

Lesson Two

The Evolution of Numbers

- I. Introductory Remarks
- II. Enumeration
- III. Numeration
- IV. Number
- V. Numeration Systems
- VI. Number Systems
 - A. The set of natural numbers
 - B. The set of whole numbers
 - C. The set of integers
 - D. The set of rational numbers
 - E. The set of real numbers
- VII. How Pythagoras encountered $\sqrt{2}$

Lesson Three

The Language of Algebra

- I. Introductory Remarks
 - A. Recall: $\mathbb{R} + \mathbb{Q} \cup \text{"I"} (\cup \text{ means "union")}$
 - B. The set of complex numbers
- II. Components of the language of Algebra
 - A. Variables
 - B. Constants
 - C. Symbols
 - D. Functions
- III. Introduction to Monomials
 - A. Examples of monomials
 - B. Definition of a monomial
 - C. Visualizing monomials
 - D. Adding/subtracting monomials
 - E. Multiplying/Dividing monomials
- IV. Introduction to Polynomials
 - A. Multiplying binomials
 - 1. The distributive law
 - 2. Factoring
 - B. Three identities to remember
 - 1. The square of a sum
 - 2. The square of a difference
 - 3. The difference of two squares

Lesson Four

Exploring Functions with the Aid of Graphing Calculators

- I. The Multiple Uses of Variables
 - A. A label in a formula
 - B. A generalized number in an identity/definition
 - C. A generalized number in a property of real numbers
 - D. An unknown in an equation to solve
 - E. An argument of a function
- II. An Introduction to the Most Frequently Used Features of the Graphing Calculator (throughout this course)
 - A. The Function Keys
 - 1. Y=
 - 2. WINDOW
 - 3. ZOOM
 - 4. TRACE
 - 5. GRAPH
 - 6. TABLE
 - 7. CALC(ULATE)
 - B. The Editing Keys
 - 1. 2nd/ALPHA
 - 2. INS(ERT) / DEL(ETE)
 - 3. Arrow Keys
 - 4. MODE
 - 5. QUIT
 - C. The Basic Statistics Keys
 - 1. STAT
 - 2. STAT PLOT
- III. Graphing Basics to Remember
 - A. Four quadrants and the respective signs of x and y .
 - B. Two axes and their names and equations

Lesson Five

Linear Functions—Introductory Explorations

- I. Introductory Remarks
 - A. Rene Descartes (1596-1650) and analytic geometry
- II. Linear Exploration #1: Even Numbers
 - A. Concrete representation with colored tiles
 - B. Derivation of algebraic formula
 - C. Input/Output numerical table
 - D. Graphical representation
 - E. General formula for the n th positive even number
- III. Linear Exploration #2: Odd Numbers
 - A. Concrete representation with colored tiles
 - B. Derivation of algebraic formula
 - C. Input/Output numerical table
 - D. Graphical representation
 - E. General formula for the n th (positive) odd number
- IV. Linear Exploration #3: Reed's Reasonable Rates
 - A. Derivation of algebraic formula
 - B. Graphical representation and exploration
- V. Linear Exploration #4: An Interest-Free Loan
 - A. Derivation of algebraic formula
 - B. Graphical representation and exploration
- VI. Lesson Recapitulation

Lesson Six

Multiple Representations of Linear Functions

- I. Introductory Remarks
 - A. The verbal/situational representation
 - B. The symbolic/algebraic representation
 - C. The numerical/tabular representation
 - D. The graphical representation
- II. Revisiting Reed's Reasonable Rates
 - A. Exploring the multiple representations with the aid of the graphing calculator
 - B. The concept of rate of change/slope
 - C. The definition of slope
 - D. The computation and interpretation of the slope
 - E. Summary: Verbal story→Algebraic equation→Numerical table→Geometric graph
- III. Revisiting the Interest-Free Loan
 - A. Exploring the multiple representations with the aid of the graphing calculator
 - B. The computation and interpretation of the slope
 - C. Summary: Verbal story→Algebraic equation→Numerical table→Geometric graph
- IV. Concluding Remarks

Lesson Seven

The Geometry of Linear Function Graphs

- I. Introductory Remarks
- II. The standard form for the equation of a linear function:

$$y=ax+b \quad \text{or}$$

$$f(x)=ax+b$$

where a and b are constants, and x and y are variables (The equivalence between $y=ax+b$ and $y=mx+b$ was noted)

- III. Graphical Exploration #1: The effect of the numerical coefficient a
 - A. If a is positive, the function increases
 - B. If a is negative, the function decreases
 - C. If a is zero, the function is constant
- IV. Graphical Exploration #2: The effect of the numerical coefficient b
 - A. If b is positive, the function's graph intersects the y -axis above the origin $(0,0)$
 - B. If b is negative, the function's graph intersects the y -axis below the origin $(0,0)$
 - C. If b is zero, the function's graph intersects the y -axis at the origin $(0,0)$
- V. Parallel Lines
- VI. Perpendicular Lines
- VII. Lesson Recapitulation
 - A. " a " is the slope or rate of change
 - B. " b " is the y -coordinate of the y -intercept $(0,b)$
 - C. Test your knowledge on a few examples

Lesson Eight

Words, Equations, Numbers, and Graphs

- I. Introductory Remarks
 - A. The “four forms” representation of a function
 - B. The importance of moving back and forth from one form of representation to another
- II. Explorations #1 and #2: NUMBERS→EQUATION→WORDS
 - A. Finding a
 - B. Finding b
 - C. Deriving the equation $y=ax+b$
 - D. Imagining a problem situation
- III. Explorations #3 and #4: GRAPH→EQUATION→WORDS
 - A. Finding a
 - B. Finding b
 - C. Deriving the equation $y=ax+b$
 - D. Imagining a problem situation
- IV. Concluding Remarks

Lesson Nine

Problem Solving with Linear Equations

- I. Introductory Remarks
 - A. A brief history of problem solving
 - B. The importance of problem solving in the evolution of algebra
- II. Problem #1: A concrete sequence of geometric figures
 - A. Derivation of the algebraic equation
 - B. Method 1: Functional exploration
 - C. Method 2: Symbolic manipulation
- III. Problem #2: A real-world problem situation
 - A. Derivation of the algebraic equation
 - B. Method 1: Functional exploration
 - C. Method 2: Symbolic manipulation
- IV. Problem #3: An abstract problem
 - A. Method 1: Functional exploration
 - B. Method 2: Symbolic manipulation
- V. Lesson Recapitulation
 - A. The connection between “a linear function” and “a linear equation”
 - B. The solution of a linear equation is unique

Lesson Ten

Modeling Real-World Data with Linear Functions

- I. Introductory Remarks
 - A. The importance of Mathematical Modeling in the algebra curriculum
 - B. What is Mathematical Modeling?
- II. Modeling Problem #1: A Scientific experiment
 - A. Demonstration
 - B. Data collection
 - C. Scatter plot
 - D. Linear regression→mathematical model
 - E. Posing investigative questions
 - F. Using the model to answer questions
- III. Modeling Problem #2: A Financial Question
 - A. Previously collected data
 - B. Scatter plot
 - C. Linear regression→mathematical model
 - D. Posing investigative questions
 - E. Using the model to answer questions
- IV. Lesson Recapitulation
 - A. The collection, representation, and analysis of data
 - B. The construction and use of a mathematical model

Lesson Eleven

Linear Functions and Geometry

- I. Introductory Remarks
- II. Investigation #1: What is the Meaning of Π (pi)?
 - A. Exploring circles
 - B. Data collection
 - C. Scatter plot
 - D. Linear regression
 - E. Questions and answers
 - F. Difference between this investigation and mathematical modeling
 - G. The story of Π (pi)
- III. Investigation #2: What is the sum of the interior angles of a polygon?
 - A. Exploring convex polygons
 - B. Data collection
 - C. Algebraic equation
 - D. Questions and answers
- IV. Lesson Recapitulation

Lesson Twelve

Quadratic Functions—Introductory Explorations I

- I. Introductory Remarks
 - A. A brief history of the quadratic equation
 - B. Connecting algebraic expressions with geometric shapes
- II. Quadratic Exploration #1: Square Numbers
 - A. Concrete representation with colored tiles
 - B. Derivation of algebraic formula
 - C. Input/output numerical table
 - D. First and second differences
 - E. Graphical representation
- III. Quadratic Exploration #2: Oblong Numbers
 - A. Concrete representation with colored tiles
 - B. Derivation of algebraic formula
 - C. Input/output numerical table
 - D. First and second differences
 - E. Graphical representation
- IV. Quadratic Exploration #3: "Star" Numbers
 - A. Concrete representation with colored tiles
 - B. Derivation of algebraic formula
 - C. Input/output numerical table
 - D. First and second differences
 - E. Graphical representation
- V. Lesson Recapitulation
 - A. The standard form for the equation of a quadratic function:
$$y = ax^2 + bx + c \quad \text{or} \quad f(x) = ax^2 + bx + c$$
 - B. The relationship between linear and quadratic functions

Lesson Thirteen

Quadratic Functions—Introductory Explorations II

- I. Introductory Remarks
- II. Quadratic Exploration #4: Gauss's Formula
 - A. The story behind this famous sequence of numbers
 - B. Concrete representation with colored multi-link cubes
 - C. Derivation of algebraic formula
 - D. Proof without words
 - E. Input/output numerical table
 - F. First and second differences
 - G. Graphical representation
- III. Quadratic Exploration #5: A Maximization Problem
 - A. Statement of the problem
 - B. Modeling the problem
 - C. Graphical representation
 - D. Derivation of algebraic formula
- IV. Lesson Recapitulation

Lesson Fourteen

The Geometry of Quadratic Function Graphs

- I. Introductory Remarks
- II. The Standard form for the Equation of a Quadratic Function:
$$y = ax^2 + bx + c \quad \text{or} \quad f(x) = ax^2 + bx + c.$$
- III. Graphical Exploration #1: The effect of the numerical coefficient a
 - A. If a is positive, the parabola opens upward; the function has an absolute minimum
 - B. If a is negative, the parabola opens downward; the function has an absolute maximum
- IV. Graphical Exploration # 2: The effect of the numerical coefficient c
 - A. $(0, c)$ is the y -intercept
- V. Graphical Exploration #3: The effect of the numerical coefficient b
- VI. Lesson Recapitulation

Lesson Fifteen

Words, Equations, Numbers, and Graphs

- I. Introductory Remarks
 - A. The method of finite differences
- II. Exploration #1: From Numbers to Equations
 - A. Deriving the algebraic equation using the method of finite differences
 - B. Questions and answers
- III. Exploration #2: From Graphs to Equations
 - A. Deriving the input/output table of numerical values
 - B. Deriving the algebraic equation using the method of finite differences
- IV. Concluding Remarks

Lesson Sixteen

Problem Solving with Quadratic Equations

- I. Introductory Remarks
- II. Problem #1: A concrete sequence of geometric figures
 - A. Functional Exploration
 - 1. Tracing the graph
 - 2. Using the input/output table
- III. Problem # 2: The Golden Ratio
 - A. Deriving the golden quadratic equation
 - B. Functional Exploration
 - 1. Tracing the graph
 - 2. Using the input/output table
 - C. A brief history of the golden ratio
- IV. Problem #3: Finding the roots of a quadratic equation
 - A. Functional Exploration
 - 1. Tracing the graph
 - B. Symbolic manipulation
- V. Lesson Recapitulation
 - A. The connection between "a quadratic function" and "a quadratic equation"
 - B. Three possible cases for the solution(s) of a quadratic equation

Lesson Seventeen

Modeling Real-World Data with Quadratic Functions

- I. Introductory Remarks
 - A. The definition of mathematical modeling revisited
- II. Modeling Problem #1: An Object in Free Fall
 - A. Previously collected data
 - B. First differences
 - C. Investigative questions
 - D. Scatter plot
 - E. Quadratic regression→mathematical model
 - F. Using the model to answer questions and make predictions
- III. Modeling Problem #2: A Theater Company's Dilemma
 - A. Previously collected data
 - B. First differences
 - C. Investigative questions
 - D. Scatter plot
 - E. Quadratic regression→mathematical model
 - F. Using the model to answer questions and make predictions
- IV. Lesson Recapitulation
 - A. The process of mathematical modeling
 - B. The importance of mathematical modeling

Lesson Eighteen
Polynomial Explorations
(Degree Greater than Two)

- I. Introductory Remarks
 - A. Linear, quadratic, and cubic functions
 - B. First-, second-, and third-degree polynomials
- II. Exploration #1: A 3-D Maximization Problem
 - A. Describing the situation
 - B. Posing the question
 - C. Visualizing the problem with the aid of a concrete model
 - D. Deriving the input/output table
 - E. Plotting the data
 - F. Finding the (functional) equation through cubic regression
 - G. Answering the question by functional exploration
 - H. Answering the question by symbolic manipulation
 - I. Checking third differences
- III. Exploration #2: A Cube-Counting Problem
 - A. Describing the situation
 - B. Posing questions
 - C. Visualizing the problem with the aid of a concrete model
 - D. Deriving the functional equations through symbol manipulation
 - E. Answering the questions by functional exploration
 - F. Multiplying out the cubic function
- IV. Lesson Recapitulation
 - A. Comparing cubic, quadratic, and linear functions

Lesson Nineteen
Rational Functions—Introductory Explorations

- I. Introductory Remarks
 - A. A new class of functions
- II. Rational Exploration #1: Varying the perimeter while keeping the area constant
 - A. Visualizing the concept with the aid of manipulatives
 - B. Describing the situation
 - C. Posing the questions
 - D. Modeling the problem with square tiles
 - E. Collecting the data
 - F. Deriving the (functional) equation by symbolic manipulation
 - G. Graphing the function
 - H. Answering the questions by functional exploration
 - I. Recapitulating Exploration #1
 - 1. The general form of a rational function
 - 2. The simplest form of a rational function
- III. Rational Exploration #2: Exploring the reciprocal function $f(x)=1/x$
 - A. Examining input/output tables
 - B. Inversely proportional quantities
 - C. Exploring the graph of $f(x)=1/x$
 - D. Recapitulating Exploration #2
 - 1. Properties of the reciprocal function
- IV. Concluding Remarks

Lesson Twenty

The Geometry of Rational Function Graphs

- I. Introductory Remarks
 - A. The connection between algebraic equation and geometric graph
- II. Graphical Exploration #1: $f(x)=1/x-2$
 - A. Graphing the function
 - B. Vertical asymptote at $x=2$
 - C. y-intercept at $(0,-1/2)$
 - D. There are no x-intercepts
 - E. There are no (absolute) maximum or minimum values of the function
 - F. Horizontal translation of $f(x)=1/x$
- III. Graphical Exploration #2: $g(x)=x-2/x+3$
 - A. Graphing the function
 - B. Vertical asymptote at $x=-3$
 - C. y-intercept at $(0,-2/3)$
 - D. x-intercept at $(2,0)$
 - E. There are no (absolute) maximum or minimum values of the function
- IV. Suggestions for Graphical Explorations #3 and #4:
 $h(x)=2/x^2$ and $i(x)=2x+4/x^2-1$
- V. Lesson Recapitulation
 - A. The general form for the equation of a rational function
 - B. Properties of rational functions

Lesson Twenty-One

Working with Rational Functions and Equations

- I. Introductory Remarks
 - A. A second look at $f(x)=1/x$
 - B. The general case: $f(x)=a/x$, where a is a constant
- II. Problem #1: A Real-World Situation
 - A. Describing the situation
 - B. Posing the questions
 - C. Deriving the equation through symbolic manipulation
 - D. Graphing the function
 - E. Answering the questions by functional exploration
 - F. Confirmation of answers by symbolic manipulation
- III. Problem #2: A 3-D Geometric Problem
 - A. Describing the situation
 - B. Posing the question
 - C. Visualizing the problem with the aid of manipulatives
 - D. Deriving the equation through symbolic manipulation
 - E. Graphing the function
 - F. Answering the question by functional exploration
 - G. Confirmation of answer by symbolic manipulation
- IV. Concluding Remarks

Lesson Twenty-Two

Exponential Functions—Introductory Explorations

- I. Introductory Remarks
 - A. A new class of functions
 - B. Multiple applications of exponential functions
- II. Exponential Exploration #1: The Mythical Story of Chess
 - A. Telling the story
 - B. Modeling the situation with the aid of manipulatives
 - C. Posing some questions
 - D. Conjecturing
 - E. Deriving the algebraic equation from the input/output table
 - F. Graphing the function
 - G. Answering questions by functional exploration
 - H. Answering questions by symbolic manipulation
- III. Exponential Exploration #2: A Paper-Folding Activity
 - A. Describing the situation
 - B. Posing the question
 - C. Conjecturing
 - D. Modeling the situation with the aid of manipulatives
 - E. Answering the question with the aid of the calculator
- IV. Suggestion for Exponential Exploration #3: The Best Choice of a Summer Job
- V. Concluding Remarks

Lesson Twenty-Three

The Geometry of Exponential Function Graphs

- I. Introductory Remarks
 - A. The connection between the algebraic equation and the geometric graph
 - B. Review of exponential rules
- II. Graphical Exploration #1: $f(x) = 2^x$
 - A. A counter example
 - B. Comparing the graphs of $f(x) = 2^x$ and $f(x) = x^2$
 - C. Other examples: $f(x) = 3^x$; $f(x) = 10^x$; $f(x) = (\frac{1}{2})^x$
 - D. x -intercepts : none
 - E. y -intercept: (0,1)
 - F. General form for the equation of the simplest exponential function:
$$f(x) = y = b^x, b > 0$$
- III. Lesson Recapitulation
 - A. Properties of $f(x) = b^x, b > 0$

Lesson Twenty-Four

Working with Exponential Functions and Equations

- I. Introductory Remarks
 - A. Modeling real-world data with exponential functions
 - B. A brief explanation of radioactivity
 - C. Explanation of general form: $f(x)=y=Ab^x$
- II. Modeling Problem #1: Radioactive Decay
 - A. Modeling the problem with the aid of manipulatives
 - B. Collecting the data
 - C. Plotting the data
 - D. Deriving the mathematical model through exponential regression
 - E. Posing some questions
 - F. Using the mathematical model to answer the questions
- III. Modeling Problem #2: Sports Analysis
 - A. Using previously collected data
 - B. Plotting the data
 - C. Deriving the mathematical model through exponential regression
 - D. Suggesting investigative questions
- IV. Concluding Remarks

Lesson Twenty-Five

Systems of Linear Functions and Equations

- I. Introductory Remarks
 - A. The meaning of a "system"
 - B. Alternate forms for the equation of a straight line
 - C. A system of two linear equations in two unknown
- II. System #1: A Real-World Situation
 - A. Describing the situation
 - B. Posing the question
 - C. Deriving the system of equations by symbolic manipulation
 - D. Graphing the system
 - E. Solving the system by functional exploration
 - F. Solving the system by substitution method
 - G. Solving the system by elimination method
- III. System #2: An abstract system of equations
 - A. Solving the system by substitution method
 - B. Solving the system by elimination method
 - C. Solving the system by functional exploration
- IV. Lesson Recapitulation
 - A. The solution of a system of two linear equations in two unknowns
 - 1. One solution: The equations are consistent and the lines intersect at one point
 - 2. No solution: The equations are inconsistent and the lines are parallel
 - 3. An infinite number of solutions: The equations are linearly dependent and the lines coincide

Lesson Twenty-Six

Using Matrices to Solve Linear Systems

- I. Introductory Remarks
 - A. A new method for solving systems of linear equations: matrices
 - B. A brief history of matrices
- II. System #1: A Fund-Raising Problem
 - A. Describing the situation
 - B. Deriving the system of equations
 - C. Introducing matrices into the system
 - D. Using matrices to solve the system
 - E. Checking the answer
- III. System #2: A System of Three Equations in Three Unknowns
 - A. Introducing matrices into the system
 - B. Using matrices to solve the system
- IV. Lesson Recapitulation
 - A. The coefficient matrix
 - B. The variable matrix
 - C. The constant matrix
 - D. $x = A^{-1} \bullet B$
- V. Concluding Remarks
 - A. Matrices: An extremely useful tool
 - B. Tools: Using versus understanding

Lesson Twenty-Seven

Systems of Functions and Equations

- I. Introductory Remarks
 - A. A variety of systems
 - B. The logical reasoning behind a system
- II. System #1: Finding the Side Length of a Square Knowing its Area
 - A. Extracting the square root
 - B. Graphing the system
- III. System #2: A Linear and a Quadratic Function
 - A. Solving the system by graphical exploration
 - B. Solving the system by symbolic manipulation
 - C. Connecting the two approaches
- IV. System #3: A Linear and a Quadratic Function
 - A. Generalizing the solution of a system consisting of one linear and one quadratic function
- V. System #4: A Linear and a Rational Function
 - A. A real-world situation
 - B. Deriving the functional equations
 - C. Solving the system by graphical exploration
- VI. System #5: A Linear and a Rational Function
 - A. A number theory problem
 - B. Deriving the functional equations
 - C. Solving the system by graphical exploration
 - D. Looking at the algebraic equation
- VII. System #6: A Quadratic and an Exponential Function
 - A. Solving the system by graphical exploration
 - B. Checking the answers
- VIII. Concluding Remarks

Lesson Twenty-Eight

Systems of Inequalities

- I. Introductory Remarks
 - A. The meaning of an inequality
 - B. Comparing equations and inequalities: similarities and differences
- II. One Linear Inequality on One Unknown
 - A. Solving the inequality by symbolic manipulation
 - B. An infinite number of solutions
- III. One Linear Inequality in Two Unknowns
 - A. Solving the inequality by graphical exploration
 - B. Generalizing the meaning of one linear inequality in two unknowns
- IV. A System of Two Linear Inequalities in Two Unknowns
 - A. Solving the system by graphical exploration
- V. Suggestion for further problems
- VI. Concluding Remarks
 - A. Beware of "one-way-mathematics"

Lesson Twenty-Nine

Iterating Functions—Looking at Functions Recursively

- I. Introductory Remarks
 - A. Recursion and iteration
 - B. Topics in discrete mathematics
 - C. Discrete versus continuous
- II. Recursion
 - A. An intuitive idea of recursion: Recursion in everyday experiences
 - 1. Counting
 - 2. Nursery rhymes / folk songs
 - 3. Pictures / labels
 - 4. Toys / Russian dolls
 - 5. Puzzles / The Tower of Hanoi
 - 6. The computer world
- III. Mathematical Examples of Recursion
 - A. Recursive definitions
 - 1. Exponents
 - 2. Factorials
 - B. Sequences
 - Recursion – iteration connection
 - C. Recurrence Relations
 - 1. First-order relations
 - 2. Revisiting the triangular numbers
 - 3. The Fibonacci numbers (a second-order relation)
 - 4. Recursive versus explicit
- IV. Concluding Remarks

Lesson Thirty

Using Iteration as a Problem Solving Tool

Notes

- I. Introductory Remarks
 - A. The importance of iteration in a technological world
 - B. The power behind fractal geometry and chaos theory
- II. Exploration #1: The Sierpinski Triangle
 - A. The Geometric approach
 - B. The Chaos game
- III. Exploration #2: The Mathematics of Finance
 - A. Simple annual interest
 - B. Buying a car
 - 1. Loan amortization
 - 2. Solving the problem through iteration
- IV. Concluding Remarks
 - A. Iteration as a powerful problem-solving tool.